FLUID FLOW THROUGH A POROUS MEDIUM USING MATLAB

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***Abstract*—A porous medium or a porous material is a material containing pores (voids). The pores are typically filled with a fluid (liquid or gas). A porous medium is characterized by its porosity, permeability, tensile strength, electrical conductivity, tortuosity. Many natural substances such as rocks and soil (e.g. aquifers, petroleum reservoirs), zeolites, biological tissues (e.g. bones, wood, cork), and man-made materials such as cements and ceramics can be considered as porous media. The flow of fluid through such media is an important process that has many applications including in inkjet printing, nuclear waste disposal and petroleum and oil industries. The fundamental law governing the flow of fluids in porous media is given by Darcy's Law. A 2-D study of fluid flow in a porous reservoir is the main objective of this study. The partial differential equations resulting from continuity and Darcy's Law are solved using a simplified Finite Volume Method known as the Two Point Flux Approximation Method (TPFA Method). The code implementation has been carried out using MATLAB. The computation has been carried out for two cases: one for a coarse grid without smoothing and the other for a fine grid with smoothing. The corresponding contours of pressure and mass fluxes have been plotted in MATLAB.**

I. Introduction

A porous medium or a porous material is a material containing pores (voids). The pores are typically filled with a fluid (liquid or gas). A porous medium is characterized by its porosity, permeability, tensile strength, electrical conductivity, tortuosity. Many natural substances such as rocks and soil (e.g. aquifers, petroleum reservoirs), zeolites, biological tissues (e.g. bones, wood, cork), and man-made materials such as cements and ceramics can be considered as porous media. Fluid flow through porous media is the manner in which fluids behave when flowing through a porous medium, for example sponge or wood, or when filtering water using sand or another porous material. The theory of porous flows has applications in inkjet printing and nuclear waste disposal technologies.

A 2-D study of fluid flow in a porous reservoir is the main objective of this study. The partial differential equations resulting from continuity and Darcy's Law are solved using a simplified Finite Volume Method known as the Two Point Flux Approximation Method (TPFA Method). The code implementation has been carried out using MATLAB.

II. Modelling the Problem

A. Governing Equations

The basic equation describing the process for the fluid flow through porous medium is the continuity equation which states that mass is conserved

$$\frac{\partial(\phi\rho)}{\partial t} + \nabla\cdot(\rho v) = q.$$

Here the source term q models sources and sinks, that is, outflow and inflow per volume.

For low flow velocities v, filtration through porous media is modeled with an empirical relation called Darcy's law. According to Darcy's law,

$$v = -\frac{\mathbf{K}}{\mu}(\nabla p + \rho g \nabla z)$$

Here K is the permeability, µ is the viscosity, g is the gravitational constant and z is the spatial coordinate in the upward vertical direction.

To solve for the pressure, Darcy's law is combined with the continuity equation. For simplicity, the porosity φ is assumed constant in time and that the flow can be adequately modelled by assuming incompressibility, i.e., constant density.

$$\nabla\cdot v_w = \nabla\cdot\left[-\frac{\mathbf{K}}{\mu_w}(\nabla p_w - \rho_w G)\right] = \frac{q_w}{\rho_w}.$$

B. Assumptions

Some of the assumptions taken into consideration while modelling the phenomenon of fluid flow through porous media are as follows.

- The flow is considered to be incompressible as the fluid is water and under the given conditions of pressure and temperature the compressibility effects of water can be neglected.
- The permeability is a cell wise constant that has to be defined by the user while calling the function. The permeability is a tensor.
- Only conditions for a source and sink are provided in order to avoid other complex effects.
- Turbulence effects are neglected.
- The flow is considered to be inviscid and wall effects are ignored.
- There is assumed to be no heat transfer phenomena taking place in the control volume.

C. Simple Finite Volume Method

In classical finite-difference methods, partial differential equations (PDEs) are approximated by replacing the partial derivatives with appropriate divided differences between point-values on a discrete set of points in the domain. Finite-volume methods, on the other hand, have a more physical motivation and are derived from conservation of (physical) quantities over cell volumes. Thus, in a finite-volume method the unknown functions are represented in terms of average values over a set of finite-volumes, over which the integrated PDE model is required to hold in an averaged sense

To derive a set of finite-volume mass-balance equations for above equation, denote by Ωi a grid cell in Ω and consider the following integral over Ωi:

$$\int_{\Omega_i} \left(\frac{q_w}{\rho_w} - \nabla \cdot v_w \right) dx = 0.$$

Invoking the divergence theorem, assuming that Vw is sufficiently smooth, we get a mass balance equation

$$\int_{\partial \Omega_i} v_w \cdot n \, d\nu = \int_{\Omega_i} \frac{q_w}{\rho_w} \, dx$$

Here n denotes the outward-pointing unit normal on ∂Ωi .

To formulate the standard two-point flux-approximation (TPFA) finite volume scheme the above equation is reformulated into

$$-\nabla \cdot \lambda \nabla u_w = \frac{q_w}{\rho_w},$$

where λ = K/µw and uw = pw +ρwgz.

The TPFA scheme uses two points, the cell-averages ui and uj, to approximate the flux

$$v_{ij} = -\int_{\gamma_{ij}} (\lambda \nabla u) \cdot n \; d\nu$$

The gradient ∇u on γij in the TPFA method is now replaced with

$$\delta u_{ij} = \frac{2(u_j - u_i)}{\Delta x_i + \Delta x_j}$$

where Δxi and Δxj denote the respective cell dimensions in the x-coordinate direction.

The following expression for Vij is obtained

$$v_{ij} = \delta u_{ij} \int_{\gamma_{ij}} \lambda d\nu = \frac{2(u_j - u_i)}{\Delta x_i + \Delta x_j} \int_{\gamma_{ij}} \lambda d\nu.$$

In the TPFA method approximation of λ on γij is done by taking a distance-weighted harmonic average of the respective directional cell permeabilities,

$$\lambda_{i,ij} = n_{ij} \cdot \lambda_i n_{ij} \text{ and } \lambda_{j,ij} = n_{ij} \cdot \lambda_j n_{ij}$$

$$\lambda_{ij} = (\Delta x_i + \Delta x_j) \left(\frac{\Delta x_i}{\lambda_{i,ij}} + \frac{\Delta x_j}{\lambda_{j,ij}} \right)^{-1}$$

The flux Vij is approximated in the TPFA method in the following way

$$v_{ij} = -|\gamma_{ij}| \lambda_{ij} \delta u_{ij} = 2|\gamma_{ij}| \left(\frac{\Delta x_i}{\lambda_{i,ij}} + \frac{\Delta x_j}{\lambda_{j,ij}} \right)^{-1} (u_i - u_j).$$

In the literature on finite-volume methods it is common to express the flux Vij in a more compact form. Terms that do not involve the cell potentials Ui are usually gathered into an interface transmissibility tij.

$$t_{ij} = 2|\gamma_{ij}| \left(\frac{\Delta x_i}{\lambda_{i,ij}} + \frac{\Delta x_j}{\lambda_{j,ij}} \right)^{-1}$$

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In the literature on finite-volume methods it is common to express the flux vij in a more compact form. Terms that do not involve the cell potentials ui are usually gathered into an interface transmissibility tij.

$$t_{ij} = 2|\gamma_{ij}| \left(\frac{\Delta x_i}{\lambda_{i,ij}} + \frac{\Delta x_j}{\lambda_{j,ij}} \right)^{-1}$$

Thus, by inserting the expression for tij into (ii), we see that the TPFA scheme for in compact form, seeks a cell-wise constant function u = {ui} that satisfies the following system of equations:

$$\sum_i t_{ij}(u_i - u_j) = \int_{\Omega_i} f \; dx, \qquad \forall \Omega_i \subset \Omega$$

This system is clearly symmetric, and a solution is, as for the continuous problem. The system is made positive definite, and symmetry is preserved, by forcing u1 = 0, for instance. That is, by adding a positive constant to the first diagonal of the matrix A = [aik], where:

$$a_{ik} = \begin{cases} \sum_j t_{ij} & \text{if } k = i, \\ -t_{ik} & \text{if } k \neq i, \end{cases}$$

The matrix A is sparse, consisting of a tridiagonal part corresponding to the x-derivative, and two off-diagonal bands corresponding to the y-derivatives.

D. MATLAB Code

A short MATLAB code for the implementation of (iii) on a uniform Cartesian grid. In this code K is a 3×Nx×Ny×Nz matrix holding the three diagonals of the tensor λ for each grid cell. The coefficient matrix A is sparse and is therefore generated with Matlab's built-in sparse matrix functions. The constant that we use to force u1 = 0 in element A(1,1) is taken as the sum of the diagonals in λ. This is done in order to control that this extra equation does not have an adverse effect on the condition number of A.

```
function [P,V]=TPFA(Grid,K,q)

Nx=Grid.Nx; Ny=Grid.Ny; Nz=Grid.Nz; N=Nx*Ny*Nz;
%fprintf('%5i\n',Nx);
%fprintf('%5i\n',Ny);
hx=Grid.hx; hy=Grid.hy; hz=Grid.hz;
L = K.^(-1);
%fprint('%8.3f\n',L);
tx = 2*hy*hz/hx;
TX = zeros(Nx+1,Ny,Nz);
ty = 2*hx*hz/hy;
TY = zeros(Nx,Ny+1,Nz);
tz = 2*hx*hy/hz;
TZ = zeros(Nx,Ny,Nz+1);
L(1,1:Nx-1,:,:)+L(1,2:Nx ,:,:);
L(2,:,1: Ny-1,:)+L(2,:,2:Ny,:);
L(3,:,:,1: Nz-1)+L(3,:,:,2:Nz);
TX(2:Nx,:,:) = tx./(L(1,1:Nx-1,:,:)+L(1,2:Nx
,:,:));
TY(:,2:Ny,:) = ty./(L(2,:,1: Ny-
1,:)+L(2,:,2:Ny,:));%fprintf('%8.2f',TY);
TZ (:,:,2: Nz) = tz./(L(3,:,:,1: Nz-
1)+L(3,:,:,2:Nz));%fprintf('%8.3f',TZ);
x1 = reshape(TX(1:Nx,:,:),N,1);
x2 = reshape(TX(2:Nx+1,:,:),N,1);
y1 = reshape(TY(:,1:Ny,:),N,1);
y2 = reshape(TY(:,2:Ny+1,:),N,1);
z1 = reshape(TZ(:,:,1:Nz),N,1);
z2 = reshape(TZ(:,:,2:Nz+1),N,1);
DiagVecs = [-z2,-y2,-x2,x1+x2+y1+y2+z1+z2,-x1,-
y1,-z1]
DiagIndx = [-Nx*Ny,-Nx,-1,0,1,Nx,Nx*Ny]
A = spdiags(DiagVecs,DiagIndx,N,N)
A(1,1) = A(1,1)+sum(Grid.K(:,1,1,1));
u = A\q
P = reshape(u,Nx,Ny,Nz)
V.x = zeros(Nx+1,Ny,Nz);
V.y = zeros(Nx,Ny+1,Nz);
```

```
V.z = zeros(Nx,Ny,Nz+1);
V.x(2:Nx ,:,:) = ( P(1:Nx-1,:,:)-
P(2:Nx,:,:)).*TX(2:Nx,:,:);
V.y (:,2: Ny,:) = (P(:,1:Ny-1,:)-
P(:,2:Ny,:)).*TY(:,2:Ny,:);
V.z (:,:,2: Nz) = (P (:,:,1: Nz-1)-
P(:,:,2:Nz)).*TZ(:,:,2:Nz);
```

III. SIMULATION CASES

A. *Coarse Grid*

Consider a homogeneous and isotropic permeability $K \equiv 1$ for all $x \in R^2$. Place an injection well at the origin and production wells at the points $(\pm 1, \pm 1)$ and specify no-flow conditions at the boundaries.

These boundary conditions give the same flow as if we repeated the five-spot well pattern to infinity in every direction. The flow in the five-spot is symmetric about both the coordinate axes. We can therefore reduce the computational domain to a quarter, and use e.g., the unit box $\Omega = [0, 1]^2$.

The corresponding problem is called a quarter-five spot problem, and is a standard test-case for numerical methods in reservoir simulation.

The pressure P is computed by the following lines for a 8×8 grid:

```
Grid.Nx=8; Grid.hx=1/Grid.Nx;
Grid.Ny=8; Grid.hy=1/Grid.Ny;
Grid.Nz=1; Grid.hz=1/Grid.Nz;
Grid.K=ones(3,Grid.Nx,Grid.Ny);
N=Grid.Nx*Grid.Ny*Grid.Nz; q=zeros(N,1); q([1
N])=[1 -1];
[P,V]=TPFA(Grid,Grid.K,q);
%contourf(P,50)
grid on
%contourf(V.x,50)
hold
contourf(V.y,50)
```

B. *Fine Grid*

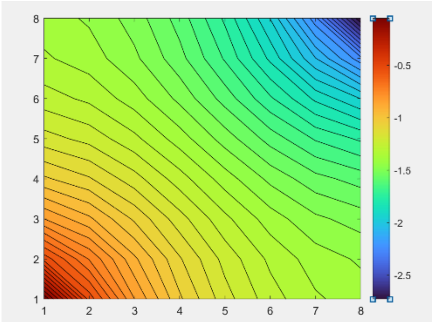
The number of grid-cells are increased in each direction from eight to 32 and consider a slightly more realistic permeability field obtained from a log-normal distribution.
3-point smoothing is applied for the given mesh to "smoothen" the discontinuities in the values of pressure gradients, velocity and flux. These arise due to the values of permeability (K) tensor being different for the cells as determined by the line for Grid.k as shown below. The pressure P is computed by the following lines for a 32 x 32 grid:

```
Grid.Nx=32; Grid.hx=1/Grid.Nx;
Grid.Ny=32; Grid.hy=1/Grid.Ny;
Grid.Nz=1; Grid.hz=1/Grid.Nz;
Grid.K=exp(5*smooth3(smooth3(randn(3,Grid.Nx,
Grid.Ny))));
N=Grid.Nx*Grid.Ny*Grid.Nz; q=zeros(N,1); q([1
N])=[1 -1];
[P,V]=TPFA(Grid,Grid.K,q);
grid on
%contourf(P,50)
hold
%contourf(V.x,50)
contourf(V.y,50)
```

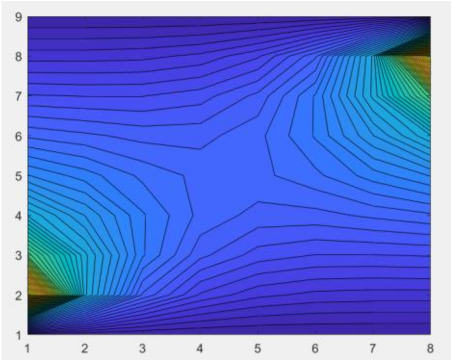
IV. RESULTS

The results of the pressure gradient, Vx and Vy contours for the coarse grid are as follows:

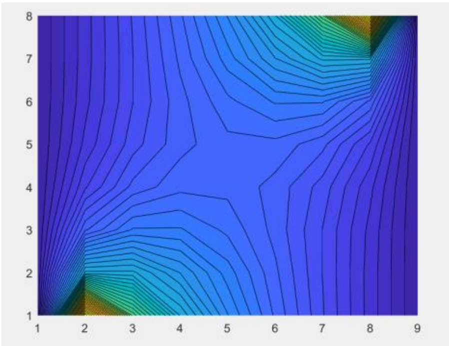
PRESSURE CONTOURS



V.x CONTOURS

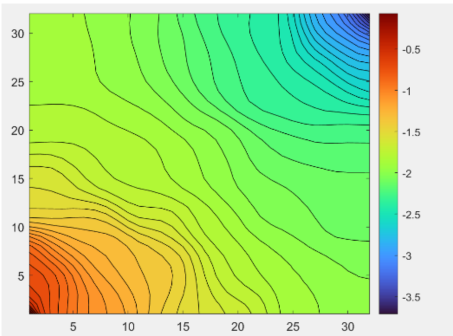


V.y CONTOURS

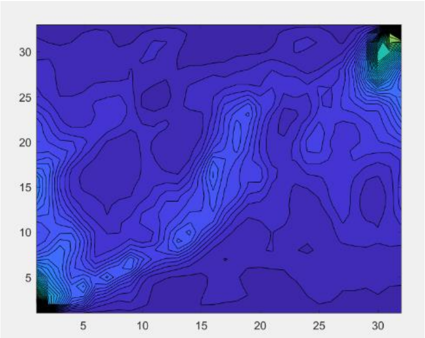


The results of the pressure gradient, Vx and Vy contours for the fine grid are as follows:

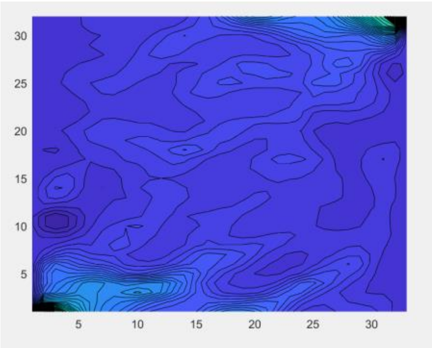
PRESSURE CONTOURS



V.x CONTOURS



V.y CONTOURS



As particles flow in directions of decreasing pressure gradient, the pressure decays from the injector in the lower-left to the producer in the upper-right corner. This is true for both cases (coarse and fine grids).

For the coarse grid case, since the permeability tensor is constant cell-wise it can be seen that the fluxes do not vary much within the domain. There is a large variation at the lower-left and upper-right corners due to the presence of the source and sink terms.

For the fine grid case, it can be seen that there are several local changes in the value of fluxes. This is because the value of the permeability tensor varies for the different cells. Hence, the flux is higher locally where the permeability is more and the flux is less where the permeability is less.

However, the overall trend of fluid flow in the direction of decreasing pressure gradient (from source to sink) is still valid in this case as can be seen in the contour plot of pressure.

V. CONCLUSION

1. An introductory modelling approach to porous flow media has been conducted.

2. The important parameters related to this have been discussed.

3. The basic PDE's have been discussed by considering continuity equation coupled with Darcy's Law for flow in porous media.

4.A MATLAB code has been presented for solving incompressible fluid flow in porous media using the Two Point flux approximation method (Simple Finite Volume method).

5.The resulting contours of pressure have been plotted for two cases with coarse and fine grids with smoothing applied to it.

VI. FUTURE SCOPE

- More emphasis can be placed on fluid properties
- Advanced approaches like MPFA can be applied.
- Both orthogonal and hexagonal grids can be considered for analysis.
- Spectral and FEM methods can be used

REFERENCES

[1] Jørg E. Aarnes, Tore Gimse, and Knut–Andreas Lie: `An Introduction to the Numerics of Flow in Porous Media using Matlab', Geometric Modelling, Numerical Simulation, and Optimization:,Applied Mathematics at SINTEF, 2007.

[2] L. Durlofsky and J. F. Brady: `Analysis of the Brinkman equation as a model for flow in porous media', The Physics of Fluids 30, 3329 (1987).

[3] K. B. Nakshatrala, S. H. S. Joodat and R. Ballarin: `Modeling Flow in Porous Media With Double Porosity/Permeability: Mathematical Model, Properties, and Analytical Solutions', April 2018, Journal of Applied Mechanics 85(8)